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Upper secondary students' ways of operationalizing the 'for all' statement in examining differentiability of a function using CAS

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ABSTRACT

Mathematical statements involving logical quantifiers are essential in calculus and for mathematical thinking. At upper secondary level mathematics, students are confronted with the universal quantifier in the definition of differentiability of a function, going from pointwise to global (or piecewise) considerations. Based on empirical cases of students in Danish upper secondary school working with tasks on differentiability, the paper addresses students' ways of operationalizing the 'for all' statement of differentiability when using Computer Algebra Systems (CAS). The analyses of the cases illustrate three different operationalizations, which build on some of the same interpretations of the universal quantifier but turned into actions in different ways. Applying instrumental genesis together with Vergnaud's notion of scheme, the analyses illustrate how students' anticipations of dealing with 'for all' differ from their operationalizations when transitioning to instrumented techniques. This is important to take into consideration when teaching 'for all', designing tasks and selecting if, for what and how CAS should be applied in these settings.

1. Introduction

Mathematical statements involving multiple quantifiers play an essential role in calculus and are important elements of mathematical thinking, not least in relation to reasoning and proof (e.g., Epp, 2009; Selden & Selden, 1995). Several studies investigate students' interpretations of quantifiers at undergraduate level (Piatek-Jimenez, 2010; Sellers, 2018). However, students already meet quantified statements in upper secondary school in relation to concepts of calculus. In this context, the quantifiers are often expressed in terms of 'for all' and 'there exists' rather than through the symbolic representations ($\forall$ and $\exists$) (Schüler-Meyer, 2022). Thus, we find it interesting to study upper secondary students' interpretations and uses of quantifiers in one of their first encounters with them.

In Danish upper secondary school, the concepts of limit and continuity are often introduced in relation to the concepts of differentiability and the derivative rather than treated independently. It is often in the context of differentiability that students meet quantifiers more explicitly for the first time. In mathematics education research, students' conceptualizations and misunderstandings of concepts involved in differential calculus such as limit (e.g., Cornu, 2002; Roh, 2010) and tangent and tangent slope (e.g., Vincent et al., 2015; Vinner, 2002) are well-researched. Another issue is the conceptual difference of the derivative at a point and the derivative function, which is easily glossed over by students (Zandieh, 2000). In the abstraction of going from the derivative at a point to differentiability as a property of a function the universal quantifier and the existential quantifier— in case of non-differentiability—are

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essential. This shift rests heavily on generalization from pointwise considerations to global (or piecewise) considerations of the function. Pointwise differentiability is related to the concept of limit, in terms of '*if* $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \to A$*, then f is differentiable for* x_0 *with the derivative* $A = f'(x_0)$'. Whereas the global (or piecewise) considerations explicitly concern the universal quantifier when extending differentiability at a point to hold *for all* $x_0 \in I$*, where* $I \subseteq dom(f)$.

An issue of dealing with the universal quantifier is that some students consider multiple examples to comprise sufficient justification for a 'for all' statement (Barkai et al., 2002; Harel & Sowder, 2007; Sellers et al., 2021). The easiness of generating and investigating examples of differentiability and non-differentiability pointwise using CAS (Computer Algebra Systems) and DGE (Dynamic Geometry Environments) potentially reinforces this misconception. In general, the use of CAS and DGE may strongly influence students' computing and algebra skills as well as problem solving. This has been addressed from the perspective of a 'lever potential' (Dreyfus, 1994) allowing students to focus conceptually while not thinking about technically demanding computations. It has been discussed what new praxeologies emerge as a response to such tools (Lagrange, 2005) as well as the way in which, particularly, DGE support the embodied nature of mathematical thinking (Misfeldt et al., 2022; Shvarts et al., 2021). Furthermore, the feedback from CAS can challenge students' prior conceptions and anticipations such that the students must reconcile these (Fonger, 2018; Kieran & Drijvers, 2006). The perspective of instrumental genesis (e.g., Guin & Trouche, 1998) focuses on users' interaction with a tool and how this influences the mathematical thinking. Thus, digital technologies influence algebraic and computational aspects of mathematical work—in both positive and negative ways (Jankvist & Misfeldt, 2015; Winsløw, 2003). Naturally, when applying digital technologies for the work with quantifiers, these tools influence the interpretations of these quantifiers as well.

Examples of a given concept or relation, for instance generated through CAS and DGE, can be illustrative and represent cases of a general definition (Bardelle & Ferrari, 2011). On the other hand, definitions may serve as a test that examples much pass. When a "new definition is introduced, one can introduce a range of examples, phrasing each as a question" (Epp, 2009, p. 4). To generalize from multiple examples, students must reflect on them as a sequence and relate them to the underlying definition (Watson & Mason, 2006). Yet, this reveals an epistemic gap between working with the single examples using CAS and grasping the general relations presented by the definition. To understand the universality of a definition, students need an operational understanding of the symbols and the formal wording, meaning that they need to translate the formal wording into operational terms (Epp, 2009). The present study addresses the role of CAS for upper secondary students' operationalizations of formal wording of the definition of differentiability of a function.

2. Theoretical background

To address the operationalization of mathematical wording we apply Vergnaud's (2009) notions of scheme which is part of what he terms the operational form of knowledge. We outline the notion of scheme and its essential role for instrumental genesis (e.g., Drijvers et al., 2013). Afterwards we present a literature review on students' use and interpretations of quantified statements to narrow down the focus of this paper and present the research question.

2.1. *The notion of scheme and instrumental genesis*

Vergnaud's (2009) operational form of knowledge enables students to carry out mathematics through cognitive schemes for actions. The operational form of knowledge is a counterpart to enunciation of mathematics and the ability to understand and formulate mathematical sentences (Vergnaud, 2009). Thus, interpreting and formulating universal quantifier statements relies on situations that offer possibilities to develop schemes for acting with universal quantifier statements. A scheme contains rules and procedures developed from experience with previously mastered situations, which offer possibilities to adapt to new situations. Hence, a scheme enables a student to act in a given situation, which in mathematics can be thought of as working with a certain task. "On the one hand, a scheme is the invariant organization of activity for a certain class of situations. On the other hand, its analytical definition must contain open concepts and possibilities of inference" (Vergnaud, 2009, p. 88). A scheme, i.e., the invariant organization, consists of four

Table 1
The four aspects of a scheme.

	The intentional aspect	The generative aspect	The epistemic aspect	The computational aspect
Description of the aspect	What we expect and want to achieve in the given situation.	Rules or procedures we know to get started, to seek and gather information and control.	Propositions and concepts we know and find relevant to the situation.	What we can infer from our activities and results.
Elements of the aspect	Goals, subgoals, and anticipations.	Rules-of-action	Operational invariants: theorems-in-action and concepts-in-action.	Possibilities of inferences.
Description of the elements	A goal can simply just be to solve a given task. The anticipation of the given situation or task also influences what the expectations of the solution and how to reach it.	For the individual, the rules-of-action are appropriate and efficient strategies based on the variables of the situation and rely on the operational invariants involved in the *epistemic aspect.*	Theorem-in-actions are propositions held to be true by the individual upon which one can infer appropriate goals and rules. Concept-in-actions are concepts held to be relevant for the given situation used to categorize and select information.	Every activity involves inference and computation of generating goals, subgoals and rules.

aspects: the intentional, the generative, the epistemic, and the computational aspect (Vergnaud, 1998a, 2009). Each aspect of a scheme comprises some elements, which are the analytical tools for understanding a specific scheme for a given situation, elaborated in Table 1 based on Vergnaud's (1998a, 2009) descriptions.

In a scheme for a certain situation, the rules-of-action of the generative aspect and the goals and expectations of the intentional aspect build on inferences of the computational aspect, based on the operational invariants of the epistemic aspect, as illustrated in the left-hand side of Fig. 1.

The notion of scheme plays an essential role when studying students' instrumental genesis, i.e., the process of working with and getting to know a digital tool. In instrumental genesis, a given digital tool is at first an artefact, a material or symbolic object on which an individual builds an instrument. For the individual, an instrument is part artefact and part cognitive schemes (Artigue, 2002). Hence, instrumental genesis involves the construction of cognitive schemes for actions with the artefact, by which the artefact becomes an instrument for certain situations (Drijvers et al., 2013). In a practical way, Drijvers et al. (2013) distinguish the cognitive schemes from the actual techniques, being the observable gestures carried out using the artefact. A technique is the way in which a task is solved and is called an instrumented technique when working with CAS in contrast to a paper and pencil technique (Artigue, 2002). The schemes are the cognitive foundation for the techniques and are only accessible through the techniques (Drijvers et al., 2013). Fig. 1 illustrates the interaction between the user and the artefact by focusing on the links between the elements of a scheme and the execution of these in the techniques.

An important aspect of instrumental genesis is the two-way interaction between the user and the tool (Trouche, 2005). The user's operational invariants come from previous experience and infer the user's goals and rules-of-action, which then impact the user's actions with the tool. Simultaneously, the tool's configuration and feedback create new experience and influence the operational invariant, and thereby the inferences, goals, and rules-of-actions. This means that students make use of the computer in ways that make sense for them to work with the quantifiers, but the computer also influences the waysstudents then interpret the quantifiers.

The four aspects of a scheme contribute with a didactical and analytical system that resembles the structure of mathematics (Ahl & Helenius, 2018). This means that goals, rules, operational invariants, and inferences of a scheme serve as mathematically legitimate goals, concepts, theorems, and inferences, despite these may not be mathematically correct (Vergnaud, 1998b). The system is didactical because it can help design and plan tasks and teaching, and it is analytical because it can help analyze students' mathematical actions related to a given situation. Similar holds for instrumental genesis, as it provides an analytic perspective on students' interaction with digital technologies as part of their mathematical thinking (e.g., Kieran & Drijvers, 2006). The scheme-technique duality can help to consider the intended techniques and schemes with the tasks as well as to analyze of students' actual techniques and schemes and in that way elaborate the student-CAS interaction (Drijvers et al., 2013).

2.2. Research on quantified statements in mathematics education

There are several types of quantified statements in mathematics (Piatek-Jimenez, 2010; Sellers et al., 2021) and many of them contain the universal quantifier either explicit or implicit. For instance, the universal quantifier is often implicit in conditional statements (Connor et al., 2007; Healy & Hoyles., 2000) as well as in definitions, since definitions often hold a bi-conditional structure (Epp, 2009). Both implications and biimplications holds 'for all' of the specified object in the hypothesis of the statement. For example, the simple statement that an even number n is divisible by 2 can be written formally as "n is even if, and only if, n equals twice some integer" (Epp, 2009, p. 3), which also means that this holds for all even numbers. Students must learn to interpret correctly these hidden structures (Shipman, 2016). In general, issues with mathematical sentences and formalism arise because the conventions of mathematical language often differ from those of colloquial language (e.g., Bardelle, 2011; Dubinsky & Yiparaki, 2000). For instance, students' negations of single quantified 'for all' statements such as "all men are mortal" can be either "some men are mortal" or "all men are not mortal" (Ye & Czarnocha, 2012). Whereas in mathematics, the negation would be "some men are not mortal." Issues with negating universal quantified statements and the relation to the existential quantifier also apply for contraposition of conditional statements (Lin et al., 2003). The approach of eliminating counterexamples as part of contrapositive reasoning builds on the statement that "the claim is true only if no counterexamples exist" (Yopp, 2020, p. 2). Still, some students find it difficult to believe that one counterexample is sufficient to infer a 'for all' statement to be false (Balacheff, 1986), as well as one example is enough to verify an existence statement (Tirosh & Vinner, 2004).

Students often misinterpret mathematical statements which lead to misconceptions of mathematical reasoning and proofs (Epp, 2009; Sellers, 2018). Dubinsky (1997) found that the mathematical programming language ISETL could help students work and

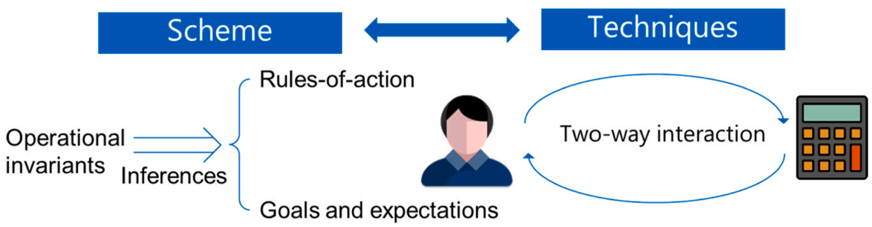

Fig. 1. An elaboration of the scheme-technique duality of the interaction between user and artefact in instrumental genesis with the relations between the four aspects of a scheme (inspired from Vergnaud, 1998b, p. 180 and Pedersen & Jankvist, 2021).

understand formal quantification. Later, CAS and DGE have shown potential with easy access to changing conditions and generating examples (Connor et al., 2007; Jankvist et al., 2019). Still, limitation can be that multiple examples are not enough to justify a universal statement. On the other hand, the tools can be useful for detecting and eliminating counterexamples in cases of existence statements, including negations of universal statements.

Many studies on students' work with quantifiers analyze students' interpretations and misconceptions (e.g., Piatek-Jimenez, 2010; Ye & Czarnocha, 2012). Other studies focus on cognitive processes leading to misconceptions. Dubinsky (1997) addressed the learning of quantified statements by applying the cognitive perspective of Action, Process, and Object (from the APOS-theory, e.g., Asiala et al., 1997). Sellers (2020) focused on students' mental actions from a constructivist perspective to analyze meanings of the individual quantified variables in complex statement, such as the Intermediate Value Theorem: *Suppose that f is a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b), such that $f(c) = N$* (Sellers et al., 2021, p. 6). A total of five student meanings of quantified variables (MQs) were identified as well as no quantification (NQ) where students did not allocate any meaning to the quantifier of a given variable (Sellers, 2020; Sellers et al., 2021).

MQ1:Checking if the predicate holds for at least one value of x in X.

MQ2:Checking if the predicate holds for exactly one value of x in X.

MQ3:Checking if the predicate holds for all values of x in X.

MQ4:Checking if the predicate holds for spontaneously chosen values of x in X.

MQ5: Checking if the predicate holds for a set-wise collection of X.

The authors found that the students used MQ1–MQ5 as well as NQ for a variety of quantifier variables. For instance, MQ1 was used for a variable quantified as 'for all' and not only as existence. The 'for all'-phrase may construct a collective meaning of the whole set of X (Dubinsky & Yiparaki, 2000; Vroom, 2022), like the MQ5. On the other hand, 'for every' seems to address each case x of the whole X individually (Vroom, 2022). Thus, the wording of the quantifiers can give rise to different meanings. The MQs include mental acts of quantifying a variable, which "involves mentally searching for (or anticipating searching for) a number of, or quantity of, values of a variable in his/her domain of discourse that satisfy the predicate in a given statement" (Sellers et al., 2021, p. 3). The students' anticipations thereby relate to their goals and expectations of the given situation, based on their interpretations of the statement. Thus, the MQs can help analyzing the schemes for students' operationalizations of quantified statements.

Most of the studies on students' work with mathematical quantifiers related to calculus concern undergraduate or teacher education. This study considers upper secondary students' work with the explicitly quantified statement, the definition of differentiability of a function. We are interested in student operationalizations of this statement and how these relate to their interpretations of the statement, thus their cognitive processes of engaging with the universal quantifier. Therefore, we see a potential for the application of Vergnaud's (1998a, 1998b, 2009) notion of a scheme, which, to our knowledge, has not previously been applied to the mathematical practices of dealing with quantified statements.[1] We focus on students' operationalizations, based on their somewhat tacit interpretations, anticipations and goals and analyze the students' actions using CAS and the cognitive schemes behind these actions to address the following question:

In what ways do upper secondary students operationalize the 'for all' statement included in the definition of differentiability of a function when using CAS?

We approach the question through the development of teaching sequence with tasks on differentiability of a function and analyses of empirical cases where students work with these tasks while consulting CAS. The theoretical perspectives of instrumental genesis (Guin & Trouche, 1998; Drijvers et al., 2013) and schemes (Vergnaud, 2009) play two roles in this study. First, they serve as a foundation for the intensions with the task design, and second, as the framework for analyzing the empirical data.

3. Method

The data presented in this paper stems from two 90-minutes lessons on differentiability in an upper secondary school in Denmark. The two lessons were conducted as part of a larger research project based on a design research approach (e.g., Bakker, 2018). This section presents the methods for the task design and the data collection, but first we account for the Danish educational context in which the lessons were developed and conducted.

3.1. Educational context and setting

The curricula for Danish upper secondary mathematics programs rely on the Danish framework for mathematical competencies (Kompetencer og Matematiklæring, KOM). The so-called KOM framework describes mathematical mastery through eight distinct, yet mutually interwoven mathematical competencies (Niss & Højgaard, 2019). These are mathematical thinking, problem handling, modeling, reasoning, representation, symbols and formalism, communication, and aids and tools competencies. Dealing with mathematical statements and quantifiers is explicitly part of the mathematical thinking competency. The competency encompasses the ability to distinguish between different types and roles of mathematical statements, such as definitions and theorems, but also to navigate with logical connectives and quantifiers in such statements. Moreover, it concerns the ability to relate to the varying scope of a mathematical concept within different contexts. For instance, to be able to extend the concept of differentiability from pointwise to

[1] Vergnaud's notion of scheme and the notion of Schema in the APOS theory both build on Piaget's work, but the two notions differ significantly.

global (or piecewise) considerations. It appears more difficult to assess students' possession and development of the mathematical thinking competency than the other competencies. For instance, the thinking competency was the only competency not being assessed as part of PISA (Programme for International Student Assessment) back when KOM's mathematical competencies laid the foundation for PISA's mathematical framework (e.g., Stacey & Turner, 2015). In the Danish mathematics program, it is the teacher's job to connect the mathematical competencies with the subject matter. The larger project focused on upper secondary mathematics students' mathematical thinking competency as defined in KOM, in interplay with the use of CAS (Pedersen, 2024). This paper focuses on the specific part of dealing with mathematical statements and quantifiers, particularly the universal quantifier as part of the definition of differentiability of a function. Because, the definition of differentiability is a biconditional mathematical statement, including the 'for all' quantifier.

The Danish mathematics programs today rest heavily on the use of digital tools. In the upper secondary level this particularly concerns CAS, which have advantages but certainly also introduce new challenges (e.g., Jankvist & Misfeldt, 2015). Danish upper secondary school takes three years, equaling grades 10–12 and the upper secondary mathematics courses cover all topics, such as algebra, geometry and calculus. Students typically have mathematics for two years, unless they have advanced mathematics, which is three years. A core topic in the second year is differential calculus with its central statement being the definition of differentiability. The presented tasks were designed for an advanced mathematics class in their second year (grade 11, age 17). The class had been working with CAS, particularly TI-nspire CAS, as an integrated part of their mathematics classes since their first year of upper secondary school.

3.2. Tasks on differentiability combining logical statements and CAS

We designed the tasks for the lessons with the aim of combining the definition of differentiability with the use of CAS. The notions of the scheme-technique duality and the four aspects of a scheme were applied to design the tasks with the intention to present situations with the 'for all' statement of differentiability of a function. The theoretical framework provided a terminology to formulate CAS' role in the task as well as for the a priori analysis of the schemes. This is elaborated in detail in section 3.3 'The rationale behind the task design'. The first author, who as a previous upper secondary mathematics teacher had worked closely together with the teacher of the class involved, designed and formulated a first draft of the tasks. The teacher then accomplished these tasks to suggest adjustments and reformulations, which were made such that the teacher could use the material as an integrated part of the total theme on differential calculus, while the intentions of the tasks (see below) were kept. The tasks presented were part of the fourth lesson on differential calculus. Previously, the students had already worked with the given functions of Tasks 6 and 7 below using dynamic features of sliders. They had investigated the secant slopes as a numerical approximation of the tangent slope (see Pedersen, 2023; Pedersen & Jessen, preprint) as an introduction to the definition of differentiability at a point of a function in Fig. 2.

The aim of the selected tasks below was for the students to work with the logical structure of the 'for all' statement contained in the definition of differentiability of a function (the last line in Fig. 2). The definition of differentiability of a function contains the universal quantifier in a biconditional statement: *the function is differentiable if and only if it is differentiable for every value of the domain*. The tasks were thought to break down the definition focusing on the difference between "*then f is differentiable for x_0 with the derivative $A = f'(x_0)$*" and "*then f is differentiable with derivative $f'(x)$ in the entire interval I*."

The tasks were distributed online in an editable Word document with an appurtenant premade TI-nspire document, where the given functions were defined and drawn in the graphic view. The premade TI-nspire document was a pragmatic choice to exempt the students from struggling with typing in the functions as well as to avoid student errors simply being due to symbolic typos. Throughout the lessons, the students could ask their teacher and other groups as well as consult all kinds of other material, e.g., textbooks, their own notes, and internet resources. In the presented part of the task sequence, the students were introduced to the CAS command for calculating the derivative but were still to consult the graphic representation as well. The tasks are translated to English and have been rephrased such that they are readable outside the context of the entire task sequence. Now we present the tasks and afterwards the rationale behind them using the theoretical framework.

Task 6: In TI-nspire is the graph of the function f, where f is defined by

$$f(x) = \begin{cases} \frac{1}{15}x^3 - 1.2x^2 + 5.4x + 1.8, & -0.5 \le x \le 9 \\ 1.8, & 9 < x \le 15 \end{cases} \tag{1}$$

DEFINITION: Let f be a function defined on an interval I and let x_0 belong to I. If

$$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \to A \ for \ \Delta x \to 0$$

where A is a real number. Then f is said to be differentiable at x_0 with derivative $A = f'(x_0)$.

If this holds for all x_0 in I, then f is differentiable with the derivative $f'(x)$ in the entire interval I.

Fig. 2. Definition of differentiability given to the students, translated from Danish to English.

a) Use CAS to calculate the derivative for the values $x = 1$, $x = 3$ and $x = 9$ and compare the results with your graphic examination in Task 4.
b) Assess whether the function is differentiable in the interval (–0.5, 15).

Task 7: In TI-nspire is the graph of the function f, where f is defined by

$$f(x) = \begin{cases} \frac{1}{12}x^2 - \frac{5}{3}x + \frac{28}{3}, & -1 \le x \le 10 \\ -0.1x + 2, & 10 < x \le 15 \end{cases} \tag{2}$$

a) Use CAS to calculate the derivative for the values $x = 2$ and $x = 10$ and compare the results with your graphic examination in Task 5.
b) Assess whether the function is differentiable in the interval $(-1, 15)$.

Task 8: In TI-nspire is the graph of the function f, where f is defined by

$$f(x) = \begin{cases} 4x + 2, & -1 \le x \le 0 \\ -0.5x^2 + 4x + 2, & 0 < x \le 4 \\ 0.1x + 9.6, & 4 < x \le 10 \end{cases} \tag{3}$$

a) Examine whether the function is differentiable.

3.3. *The rationale behind the task design*

Since the tasks were part of a sequence with focus on mathematical thinking competency, not all the presented tasks were directly targeted towards the considerations of the ‘for all’ statement in the definition, which is the focal point for this paper. The main tasks of interest for this focus are Tasks 6b, 7b and 8a. Tasks 6a and 7a serve as auxiliary tasks with answers that help solve the other tasks.

Tasks 6 and 7 build upon the same aims and intentions. The students were introduced to the specific CAS command for determining the derivative of a function for a single value $\left[\frac{d}{dx}f(x)\middle| x = x_0\right]$ referred to as the derivative-command. The purpose of the task design was for the students to construct appropriations of intended schemes, see Table 2. By applying the command for the given values in the auxiliary tasks 6a and 7a, the intention was that the derivative-command became a meaningful instrument for the students as part of their instrumented techniques for determining differentiability. By reconciling the CAS feedback with their previous work in the graphic view, the students were to extend their schemes of differentiability from focusing on one value in Tasks 6a and 7a to focusing on the entire function in Tasks 6b, 7b and 8. The intended progression is simplified in Table 2 applying the distinction between techniques as the observable part and schemes as the cognitive foundation (Drijvers et al., 2013).

This progression can seem rather linear, which most likely is not the case. The derivative-command slowly becomes an instrument for determining differentiability at point through simultaneous development of the derivative-technique and the differentiability-scheme. Below we present the intentions with the tasks to support this progression based on Vergnaud’s (2009) four aspects as guidelines.

The intentional aspect involves the intended goals and anticipation of a scheme. Our goal was for the students to develop the differentiability-scheme and its extension. This built on the anticipation that the extension of their differentiability-scheme was possible by using the derivative-technique for single values to tasks that focus on intervals.

The generative aspect involves rules-of-action, which for Tasks 6a and 7a was: *calculate the derivative (if possible) to determine if the function is differentiable for the given value*. Answering Tasks 6b and 7b can be done without CAS but was still part of the instrumental genesis as it revolved around the results of the preceding application of the derivative-technique in the auxiliary tasks 6a and 7a related to the ‘for all’ quantifier. The intention was to make the students aware of the function being piecewise defined, consisting of two elementary, differentiable functions. Relying on the rule-of-action: *check the function for critical points and determine differentiability for those x-values*, the students should invoke their differentiability-schemes with the derivative-technique and expand these to situations involving entire functions.

The epistemic aspect involves the operational invariants, upon which the other aspects are built. The intended concepts-in-action

Table 2
The intended progression developing schemes for determining differentiability applying the derivative-command.

The derivative-command is the artefact in CAS, calculating the derivative (if possible).
The derivative-technique is the application of the derivative-command for calculating the derivative (if possible) for a single value.
The differentiability-scheme is the cognitive foundation consisting of the four aspects to apply the derivative-technique for determining differentiability of a function for a single value.
The extension of the differentiability-scheme is the cognitive foundation consisting of the four aspects to apply the derivative-technique for determining differentiability of an entire function (possibly defined on an interval).

were 'derivative', differentiability', for all', existence' and 'critical point'. These were related in the theorems-in-action upon which the tasks were built. For Tasks 6a and 7a the theorem-in-action was '*if the derivative exists for the given value, then the function is differentiable for that given value*.' For Tasks 6b and 7b the theorem-in-action was '*if the function is differentiable for the value of the critical point(s), then the function is differentiable for all values, and thus differentiable for the given interval.*' The latter theorem-in-action builds on the logic of eliminating counterexamples (Yopp, 2020).

The computational aspect involves the possible inferences the students can make. It draws directly on the three other aspects. One inference would be *if you can find one value for which the function is not differentiable, then the function is non-differentiable.* It was then the generative aspect that guided how to detect if such a value would exist. Furthermore, an inference of this was the importance of how the function was defined, including which types of functions it consisted of; which were the critical values and boundary values; and was the function defined on a closed or open interval. These became particularly important for situations where the students should find critical values themselves. Therefore, we designed a similar Task 8 without an auxiliary task but building on the same intentions and assumptions as Tasks 6 and 7.

For Task 8, the students needed to activate those initial techniques and schemes they had developed working with Tasks 6 and 7. It was important that their input in CAS was meaningful for determining differentiability to get useful feedback from CAS that they could interpret as an answer in their solution path. Their solution paths and reflections upon these in Tasks 6 and 7 should prepare the ground for solving Task 8.

3.4. Data collection and analysis

During the lesson, the first author was present as an observer and responsible for the collection of data generated by the students. The 29 students were divided into 14 groups, and the data material consisted of the groups' written products, summing up to 89 pages including the tasks, written answers and screenshots. Each of the 14 groups were working on one computer. Screencasts of the students' work, including audio and webcam recordings, were collected to document the students' working processes (see Fig. 3) equaling approximately 2520 min of video. Table 3 provides an overview of the amount and selection of data.

The students' written responses were marked to indicate their type of answer: C for a *correct* answer with a *correct* explanation; W for a *wrong* answer or a correct result with a *wrong* explanation; A for an *answered* task where an unclear or no explanation was given to a correct result; and B for a *blank* answer, where no written answer was provided to the task. These marks were used as an indication for the further selection of relevant data. Yet, some students may have worked with the tasks without writing anything down. For the relevant tasks (presented above), this resulted in about 687 words with additional screenshots (Table 3).

Screencast sequences of the groups who had given a written answer (A, C, or W) to the tasks of interest were selected for further analysis of the student's work with quantifiers. The first author watched these excerpts of video recordings (approximately 165 min) to identify phenomena and episodes through an inductive selection, based on the students' use of CAS and explicit enunciations. This initial selection was based on comparisons between the different groups of students, illustrating either tendencies or differences in how the students were dealing with the tasks, leading to the identification of three illustrative episodes (approximately 50 min). The selection of these is argued for in more detail in the presentations of the cases in section 4 'Presentation of data and findings.'

In the three episodes, the students' visible actions on the screen and their enunciations of what they did and what they saw constituted the data for the further analysis. The perspective of instrumental genesis pointed the attention to the students' instrumented techniques using CAS, being the main factor for their operationalizations of the definition of differentiability. The four aspects of a scheme helped elaborate the operationalization and indicated the students' rationale behind them. Considering schemes as the organization of activity consisting of the four aspects makes it possible to analyze students' actions and inferences from these actions to make assumptions about the four aspects of students' schemes of specific situations (Elkjær & Hodgen, 2022).

First, we identified the students' instrumented techniques in the meaning of what procedures and commands they made use of. Some students explicitly stated their anticipations and goals of their techniques, whereas others did not. For the groups that did, it was possible to check for alignment between goals, anticipations and actions, indicating their rules-of-action to get started and generate information to work from. For the groups that did not, the students' goals, anticipations and rules-of-actions were identified from their reactions to the output of the techniques. The analysis of the students' experienced alignment or conflict between their techniques, the output of these, and their inferences from these became a key point for identifying their goals, anticipations and theorems-in-action, on which their actions and inferences were based. Hence, the techniques and enunciations provided the access to the four aspects of their schemes comprising the operationalizations of the statements as well as the challenges they experienced enroute. Additionally, research on students' interpretations of statements and quantifiers served to qualify the analysis in relation to the mathematical practice of dealing with quantifiers. For instance, meanings of quantification (Sellers et al., 2021) and the elimination of counter examples (Yopp, 2020) contributed with a terminology for the content of the four aspects of the students' schemes.

4. Presentation of data and findings

In this section, we begin by presenting an overview of the number of answers to each task (Table 4), based on the coding of the students' written answers in their Word documents as explained in section 3.4 on data collection.

The tasks, 6b, 7b, and 8a all concern considering differentiability of a function defined on an interval. The students had different approaches to operationalize the step from determining differentiability of a function for one value to determining differentiability of a function for an interval. From Tasks 6a and 7a the students were familiar with the derivative-command. Applying the derivative-technique, the students developed the differentiability-scheme to determine differentiability of the function for the given values ($x =$

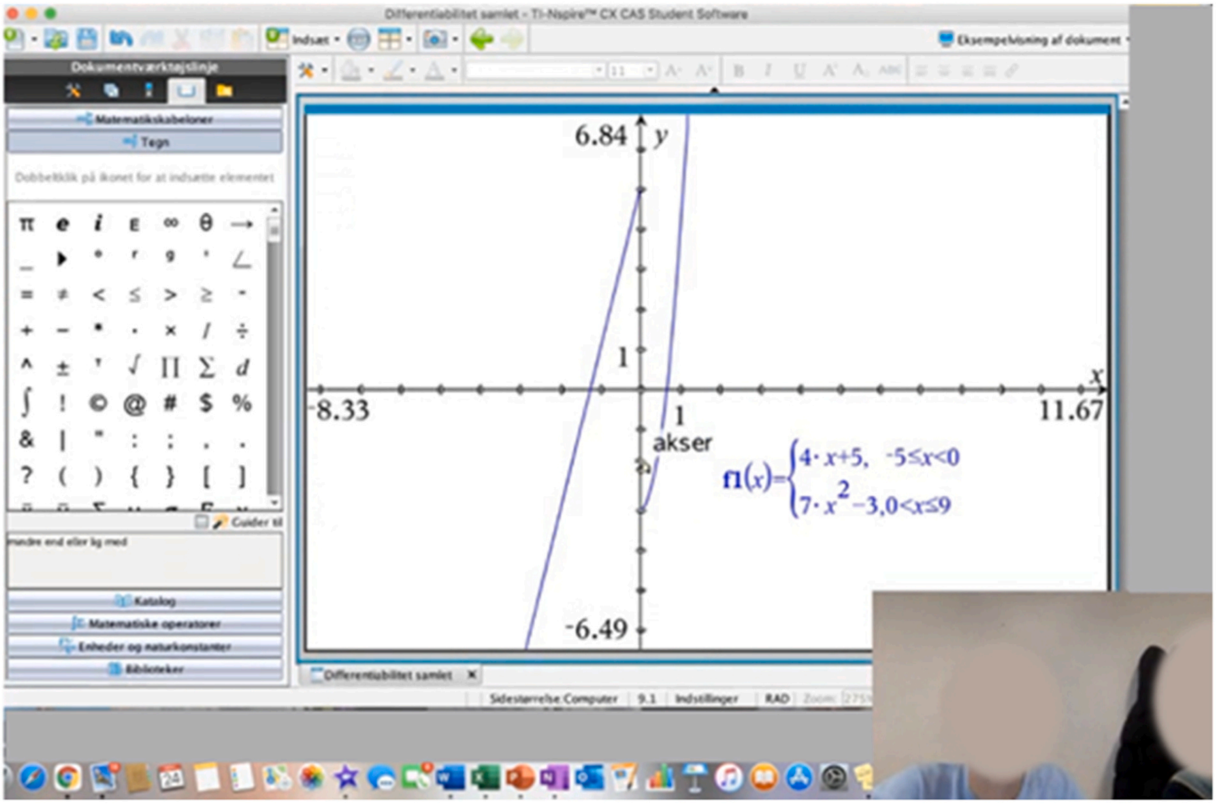

Fig. 3. Screenshot of students' screencast recording focusing on their work on the computer, here in TI-nspire, with a small webcam recording of the two students down in the right-hand corner (blurred for reasons of anonymity).

Table 3
In total, around 2520 min of screencast video together with 14 documents with a total of 3063 words of answers were collected. Of the total number of answers 687 words were coded relevant for the tasks of interests. This led to the initial analysis of a corresponding approximately 165 min of video of the students working on these tasks, which resulted in the selection of three episodes with a total amount of 50 min to be analyzed in further detail.

Collected video material	Written documents	Coded answers	Analysed video	Episodes
2520 min	14 documents (3063 words)	687 words	165 min	50 min

Table 4
Overview of the number of groups answering each of the tasks.

Task	Correct (C)	Wrong (W)	Answered (A)	Total answers	Blank (B)
6a	6	0	4	10	4
6b	1	0	9	10	4
7a	4	0	6	10	4
7b	5	1	5	11	3
8a	2	2	7	11	3

1, $x = 3$, and $x = 9$ in Task 6a, and $x = 2$ and $x = 10$ in Task 7a) as intended (cf. section 3.3 'The rationale behind the tasks'). In the following subsections, we present three cases illustrating three different ways of operationalizing the definition of differentiability of a function using CAS, building on some of the same interpretations of the universal quantifier, but executed in different ways, which we return to in the discussion. First, we account for the answers of Task 6, since Task 6b is in focus in the first case, illustrating operationalization 1. Later we will account for the answers of Tasks 7 and 8, similarly.

Out of the 14 groups, 10 groups answered both Tasks 6a and 6b. In their written work, six of the groups answered Task 6a correctly with a correct explanation and screenshots from their work using CAS. Moreover, the screencast recordings showed that also the remaining four groups reached the solution that the function was differentiable for the three given values ($x = 1$, $x = 3$, and $x = 9$) using CAS in the intended way. To Task 6b, only Group 7 had given a correct answer with an understandable explanation supporting their answer. Five of the remaining nine answers were given in a simple form like "It is differentiable in the interval" or even "That, it is", referring to the function being differentiable in the interval $(-0.5, 15)$. The other four groups gave an unclear or vague answer, e.g., Group 11, "Thus, with the help of our investigation from the previous task, we can conclude that the function is differentiable." In general, the screencast recordings revealed that most groups skipped focusing on Task 6b. They leaned against the solution of Task 6a being that the function was differentiable for all three given values and assumed the function would be differentiable in the entire interval, however, without considering the specific values or why they could make this assumption based on these values (particularly, $x = 9$). In total, two groups spent time and concentration on it: Group 7, providing a correct answer and Group 11.

4.1. Operationalization 1: using the differentiability-scheme to detect and eliminate counterexamples

One operationalization of the differentiability definition was to detect counterexamples of differentiability by detecting critical values for which the function could be non-differentiable. This way of operationalizing the definition could resemble the intended idea behind the task design, where the students negate the 'for all' statement and work from an existence statement. This means that if they could eliminate counterexamples, they could infer differentiability, or if they could find such an example, they could infer non-differentiability. However, students may not recognize this approach as the negation of a 'for all' statement to an existence statement (Bardelle, 2011). And even if they do, they may not find that such inferences can be based on only one counterexample (Balacheff, 1986; Tirosh & Vinner, 2004) . Another challenge was to detect critical values, and finally a third challenge was that some of the students based this approach on the false assumption that the function in Task 6 was non-differentiable. In such a case, the approach of eliminating counterexamples (Yopp, 2020) becomes impossible. This was the case for the student, Lou in Group 11.

Lou and her peer, Kay explicitly discussed the meaning of 'for all' and how to operationalize it in Task 6b after solving Task 6a, finding that the function was differentiable for the three given values ($x = 1$, $x = 3$, and $x = 9$), depicted in Fig. 4.

101L: Well, when it is a 'fork-function',[2], then I think that it would not be differentiable in the entire interval.

102 K:Yeees… then there is a 'sharp corner'[3];…

103 L:I guess it is differentiable in each of these intervals $[-0.5 \leq x \leq 9$ and $9 < x \leq 15]$, but I guess there is a place where it wouldn't be differentiable. So, the entire interval is not differentiable… If we try to put the point for $x = 9$, because that is where it changes.

104 K:Yes, but how do we see that it is not differentiable on this one? [referring to the derivative-command in TI-Nspire CAS] Should it say 'error' then, or something like that? … We have examined it for $x = 9$. So… Should we then examine it for…

105 L:But it was differentiable for 9, right?

106 K:Yes…

107 L:But there must be a 'sharp corner' somewhere.

The students considered the domain of the function and applied the derivative-command for $x = 15$ to which TI-nspire gave the output 'undef', which they interpreted as the function not being differentiable for $x = 15$.

108 K:Then it is because it is not differentiable, or what? Aha…

The students tried different values in the interval $(9, 15]$ and ended up trying for 14.5, 14.6, 14.7, 14.8, 14.9, and again 15, where the output changed to 'undef'. Lou still had issues with the function being differentiable at $x = 9$ and began considering the inequality signs of the symbolic expression of the function. Kay followed up on this by looking at the graph using the trace-tool. Following the graph, the students first found the point for $x = 9$. Moving on, they could not move the trace farther than $x = 14.9$ (see Fig. 5). This made the students consider the interval of examination and realized that 15 is not included in the interval of interest.

109 L:But how should you know if it is differentiable in the entire interval? Well, we cannot put them in… should we ask the teacher if there is some kind of a method?

They asked the teacher (T) who talked with them about the look of the graph as well as considering which points could be suspected of being non-differentiable.

110 L:Well, that graph we are looking at now, wasn't it 9 that was… or it was a fork-function.

111 K:Yes, that was what we thought, it was a fork-function and that hints that there is some kind of difference. Therefore, we thought if there would be… but we still got it to be differentiable.

112 T:Yes, it is still differentiable in the 'corner'.

113 L:So, it is too small of a 'corner'?

114 T:Yes, there is such a smooth transition from the first part to the second that in fact there is no 'corner', despite the change of the symbolic expression of the function.

115 L:Okay. So, it is differentiable… or it is differentiable because the transition in the 'corner' is so good… or something.

Based on this investigation, illustrated by the above transcript and description of their actions, Kay and Lou answered that the function was differentiable in Task 6b.

In the initial discussion [Lines 101–107], Lou articulated her presumptions about the situation, which indicates the intentional

[2] In Danish, Lou calls a piecewise defined function a 'gaffel-funktion', directly translated to a 'fork-function'. The term 'gaffel-funktion' is commonly used in mathematical language in Denmark, due to its divided look in the symbolic representation. However, the term 'fork-function' does not have the same indication of what it actually is, as the formal term, a piecewise defined function.

[3] The term 'sharp corner' is a translation of the Danish word 'knæk', which the students use to express a bend or a break on the graph of a function, which is still connected.

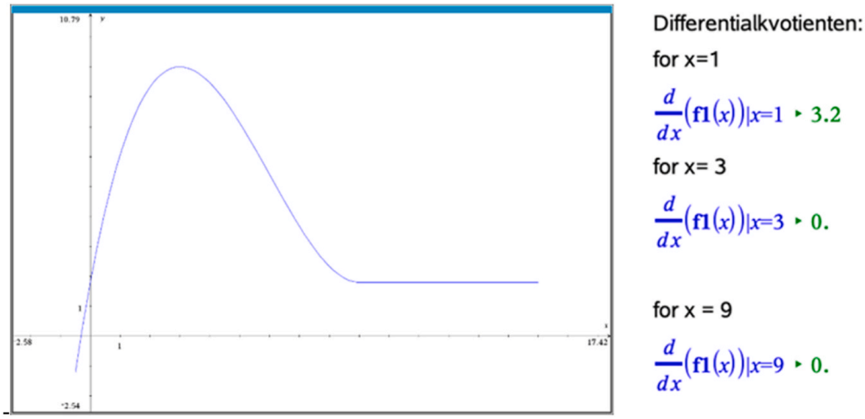

Fig. 4. The graph of the relevant function f in Task I and the calculations of the derivatives for $x = 1$, $x = 3$, and $x = 9$ in CAS (copied from the students' document).

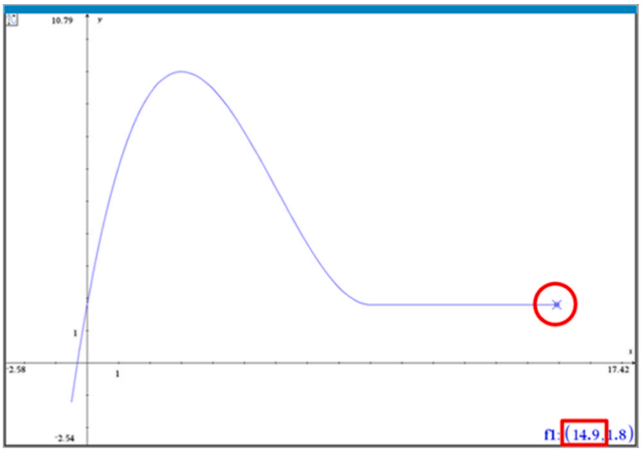

Fig. 5. An illustration of the students' trace of the graph, with the point for x = 14,9 highlighted.

aspect of her scheme for determining differentiability of a function. Her goal was to *find the counterexample* based on the anticipation that *there is one value for which the function is non-differentiable* [Line 103 and 107]. The generative aspect then involves the rule-of-action of *using the derivative-technique to find this presumed existing value*, starting with $x = 9$ [Line 103]. This led to confusion, since the students already knew that the function was differentiable for $x = 9$ [Line 105]. The explanation of this confusion is found in Lou's epistemic aspect. It includes the concept-in-action of 'a piecewise defined function', which she attached with high relevance for the situation [Lines 101 and 110]. Lou's presumptions in Lines 101 and 103 indicate two theorems-in-action. First, *a piecewise defined function is not differentiable*—a proposition, which is mathematically false. Second, *a non-differentiable function has (at least) one value for which the function is non-differentiable*. This is a true proposition, based on the logic of negating a 'for all' statement. Based on this reasoning, the computational aspect of her scheme involves the inference that *the function is non-differentiable*. In the beginning, Lou's reasoning followed the intention of Task 6b, which was to focus on the domain of the function to extend the technique and scheme with calculations from Task 6a into the context of examining the function for the interval [Line 103]. However, due to the epistemic aspect of her scheme including a false theorem-in-action, she failed to extend the differentiability-scheme to this situation, despite the accuracy of the second theorem-in-action. This implied a false anticipation of the solution and created an unsolvable task [Line 107].

The students applied their differentiability-scheme for single values together with the trace-tool to search for a value for which the function should be non-differentiable. The students could add together the provided information by CAS and the graph: such as the function was differentiable for $x = 9$ [Fig. 3]; the function was not differentiable for $x = 15$ [Line 108]; the value $x = 15$ was not of interest for the specific situation [Fig. 4]; and the function was differentiable for every value they tried within the domain. In this process, the CAS feedback influenced the students' assumptions and challenged their schemes for determining differentiability at the given interval. This made Lou question how to operationalize the 'for all' quantifier in this situation [Line 109]. The teacher confirmed that the piecewise function could be suspicious and that $x = 9$ was the value of interest, however the function was still differentiable [Lines 112 and 114]. With Lou's expression in Line 113, that the "corner is too small" she formulated the situation in terms that matched her scheme. However, she needed to discard her anticipation that the function would be non-differentiable at the 'corner' and thereby, her theorem-in-action that a piecewise defined function is differentiable. Second, the output 'undef' for $x = 15$ provided new knowledge to how CAS could help them find the assumed value for which the function would be non-differentiable [Line 108]. Hence, the derivative-technique was extended to be useful for situations of determining non-differentiability. As they could not find any other possible value than $x = 9$ that showed to be non-differentiable, Lou accepted the function was differentiable in the entire interval [Line 115].

Having spent serious time studying Task 6 in detail, Kay and Lou had no issues with solving the following Tasks 7 and 8, applying the same scheme, now to detect counterexamples. Because, in Task 7 they were given the value of $x = 10$, which was the counterexample of differentiable, and in Task 8 they had become aware of the critical values through their work with the Tasks 6 and 7. Thus, they could easily detect the counterexample of $x = 4$ as being non-differentiable. In general, ten out of 14 groups answered Task 7b. One group provided the wrong answer that the function was differentiable, despite they had answered Task 7a correctly. The remaining nine groups answered Task 7b rather quickly, and five of these with a correct written answer, as they had detected the counterexample of differentiability for $x = 10$ in Task 7a. Conversely, Task 8a caused the students that did not pay much attention to Task 6b the most troubles. Multiple students had the meaning of the universal quantifier as checking if the predicate holds for all values, identified as MQ3 by Sellers et al. (2020). For this, the students had two approaches. One was to consider the entire function at once, literally extending the derivative-technique for one value to concerning an interval, identified as operationalization 2. The other was to extend the differentiability-scheme applying the derivative-technique for every value of the domain, identified as operationalization 3.

4.2. Operationalization 2: extending the derivative-technique to check for all values at once

For Tasks 6b and 8a, students attempted to think with the tool by applying the derivative-command in TI-nspire and extending the conditions for x using the specifying command [|] on an interval instead of a single value. One of the groups using this technique was the two students of Group 7, Hal and Gia, who, as the only group, provided a correctly written answer. Prior to Task 6b, they were working on Task 6a applying the derivative-technique and developing a differentiability-scheme for determining differentiability of the given function at a point. Their calculations were like Kay and Lou's calculations in Fig. 3, except Hal and Gia reused the same command by deleting and adding specifications for each task, not leaving a screenshot in their written document. From this they moved on to Task 6b.

201 H:Determine if the function is differentiable throughout the interval [reading aloud the task].

Hal looked at the derivative-command they had been working with and deleted the specification of x after the specifying command [|].

202 H:Can't we write x equals… or greater than, uh what was that, greater than what… -1. [Hal accidentally looked at the interval for Task 7 instead of Task 6.]

203 G:No, it should not be bigger than that, because it should be the same.

204 H:Yes, the same or greater. It should be the one that… [she imitated an inequality sign with a line under with her hands. Then she began writing $x \geq -1$ in the derivative-command.] Should you then write "and"?

205 G:Yes, right.

Hal continued writing "and $x \leq 15$" in the command which led to the result with the following input and output as seen in Fig. 6.

As the two students did not know how to interpret the CAS outcome, they rejected it and asked for help to solve the task from scratch, not introducing the teacher to their attempt at using the derivative-command. Through guidance, their attention was drawn to the critical point of $x = 9$ and their previous calculation of the derivative for $x = 9$ in Task 6a, as intended with Task 6b. Compared to the other groups which did not spend an appropriate amount of time on this task, Group 7 were capable of writing a correct answer with a clear explanation (Fig. 7).

Their process of simply adjusting the specification for every value given in Task 6a led to the inference that *the derivative-command can be used to determine differentiability through specification*. This then becomes their rule-of-action on which they build their technique for Task 6b, which was to extend the use of the derivative-command. Hal's utterances of using inequality signs as well as the 'and'-command [Lines 202 and 204] indicate a goal of *checking the whole function at once*. The goal involves the interpretation of MQ3, *checking for all values of the domain*, but is operationalized as MQ5 since they were *checking for the set-wise collection* (Sellers, 2020). Thus, their operationalization of going from a pointwise to a global consideration of differentiability contains a collective meaning for the 'for all' statement (Vroom, 2022).

$$\mathbf{f1}(x) \blacktriangleright \begin{cases} \dfrac{x^3}{15} - 1.2\cdot x^2 + 5.4\cdot x + 1.8, & ^{-}0.5 \leq x \leq 9 \\ 1.8, & 9 < x \leq 15 \end{cases}$$

$$\frac{d}{dx}(\mathbf{f1}(x))|x \geq ^{-}1 \text{ and } x \leq 15 \blacktriangleright \begin{cases} 0.2\cdot x^2 - 2.4\cdot x + 5.4, & ^{-}0.5 < x \leq 9. \\ 0., & 9. < x \leq 15. \end{cases}$$

Fig. 6. Hal and Gia's application of the derivative-command on an interval in TI-nspire to solve Task 6b (this is a remake to ensure better quality).

Når vi kigger på funktionsudtrykket, kan vi se, at der sker noget ved x0=9. der ændrer udtrykket sig, men da vi regner ud at den er differentiabel i 9 vurderer vi at den må være differentiabel i hele intervallet.

Fig. 7. Hal and Gia's written answer copied from their document. The text translates to: "When we look at the function expression, we can see that something happens at x0 = 9. the expression changes, but since we calculate that it is differentiable in 9, we estimate that it must be differentiable in the entire interval.".

The students' error of typing $-1 \leq x$ was because they mistook the interval of Task 7 with that from Task 6. Their focus on interpreting and using the provided (yet wrong) interval in the task indicates that they considered the domain a relevant concept-in-action for extending this derivative-technique. In fact, it is unnecessary to specify the interval to obtain the derivative function (see Fig. 8) when the original function is defined for the interval of interest. Additionally, since $-1 < -0.5$ their error was insignificant to obtain the correct output (Fig. 6). However, the CAS outcome could not help the students further in their operationalization at this point, despite their technique is applicable for determining if the function is differentiable. TI-nspire gives the result of the derivative of the function and indicates for which values the derivative function is defined with the inequality signs, as for $x = 9$ (Figs. 6 and 8). However, at this point Hal and Gia were not paying attention to the importance of $x = 9$ for determining differentiability for the entire interval.

Similar techniques were applied by other groups to solve Task 8a, where no values to examine were given in advance. Here, TI-nspire would give the result of the derivative of the function, showing that the derivative function is not defined for $x = 4$ (Fig. 9) and therefore not differentiable for $x = 4$.

The students who tried this approach had a similar goal of *specifying the derivative-command for an interval* but struggled to decode the three split domains and to type it into TI-nspire. Unlike Hal and Gia, their capability of using CAS did not even allow them to construct a meaningful input in CAS, and these students rejected the technique before completing it. Instead, they asked the teacher for guidance, who directed them towards negating the 'for all' statement and detecting critical points. The students could then use their differentiability-scheme at a point, in the sense of applying the definition as a test (Epp, 2009), and thereby detect or eliminate counterexamples to determine differentiability of the function, as explained by Yopp (2020).

In total, 11 out of 14 groups answered Task 8a. Two of the groups gave a correct answer with the explanation that the function was not differentiable for $x = 4$ and therefore non-differentiable. Two groups provided the wrong answer, that the function was differentiable. For the remaining seven groups who had answered the task, the screencast recordings revealed that these groups also found the value $x = 4$ for which the function was non-differentiable and thereby concluded non-differentiability. Without explicitly discussing the problem of the 'for all' statement, the cases of Group 7 dealing with Task 6b and the other groups dealing with Task 8a illustrate some of the challenges of operationalizing 'for all' as MQ5 in terms of the entire interval at once when using CAS. One thing is to interpret the given domain and convert it into a meaningful CAS-input. If they then get an output, the challenge is to decode it. Both situations reveal the importance of paying attention to the inequality signs and the troubles of interpreting these, which also appears to be crucial for the following operationalization 3.

4.3. Operationalization 3: Applying the differentiability-scheme for every value

A different approach to Task 8a was to operationalize the 'for all' statement by applying the differentiability-scheme of determining differentiability at one point, which they students knew from the Tasks 6a and 7a. One of groups applying this operationalization was Group 5 of the two students Em and Fay.

301 F:"Examine if the function is differentiable." [Reading the task aloud.] The function. The entire function?

302 E:Nah… that's a bit difficult.

Fay turned to TI-nspire CAS, typed in the command $f1(x)$ such that TI-nspire gave the symbolic expression, as it was already defined in the program. Then Fay went back to the task.

303 F:Okay, how do we examine this? Should we try this one? [She dragged the derivative template into the notes in TI-nspire and wrote $\frac{d}{dx}f1(x)|x = 1$ to which TI-nspire gave the output 3.]

304 E:Why do you say $x = 1$?

305 F:Because then we can see that it is differentiable at 1 for example. Then we can just put in all … [started laughing].

306 E:Okay with all points? All infinity points [laughing].

307 F:No, I do not know how to do it… We could also write in those values, x is that one, that one and that one [pointing at the three lines defining the domain of the function].

308 E:Arh, so if you write -1. That is the least it can be.

Fay wrote $\frac{d}{dx}f1(x)|x = -1$ into TI-nspire to which the output was 'undef'. From this, they inferred that the function was not

$$\frac{d}{dx}(\mathrm{f1}(x)) \blacktriangleright \begin{cases} 0.2\cdot x^2-2.4\cdot x+5.4, & ^-0.5<x\leq 9. \\ 0., & 9.<x<15. \end{cases}$$

Fig. 8. Application of the derivative-command in Task 6b without specification of an interval.

$$f1(x) \blacktriangleright \begin{cases} 4\cdot x+2, & -1\le x\le 0 \\ -0.5\cdot x^2+4\cdot x+2, & 0<x\le 4 \\ 0.1\cdot x+9.6, & 4<x\le 10 \end{cases}$$

$$\frac{d}{dx}(f1(x)) \blacktriangleright \begin{cases} 4., & -1.<x\le 0. \\ 4.-x, & 0.<x<4. \\ 0.1, & 4.<x<10. \end{cases}$$

Fig. 9. An illustration of how to find the derivative of the function in Task 8a.

differentiable for $x = -1$. They secured themselves that -1 was included in the domain by looking at the inequality signs of the defined function. Then they tried with $x = 10$, the other end value of the domain, where the output also was 'undef' and with $x = 3$ with the output 1.

309 F:Okay. How exciting… I'll put this in. [Copied her screenshot to Word] Well, do you think that is correct?

310 E:Ya… or well, it is correct, it is more if it is the solution.

311 F:Should we ask… nah, let's move on.

Fay took a screenshot of the calculations in CAS for the two values, where the function was not differentiable and copied it to their worksheet in Word. To this she added that the function is not differentiable (Fig. 10).

The initial question of determining differentiability of an entire function in Task 8a together with the activity of calculating the derivative for $x = 1$ indicate that the generative aspect of Fay's scheme contains a rule-of-action to apply the derivative-technique. With Line 305, Fay was anticipating *checking all values in the domain*, aligning with Sellers et al.'s (2021) MQ3, which was the same anticipation as seen in Group 5 with Hal and Gia. However, it seems that the constraint of not being able to check for all values led them to a rule-of-action based on MQ4, *checking for a spontaneous number of values*. Thus, the students' operationalization of 'for all' may differ, as they lean on two different meanings of quantifications (here MQ4 and MQ5), which is related to what is thought and what is possible [Lines 301–307]. As Fay expressed the goal of determining differentiability for every single value [Line 305] together with Em questioning her [Line 306], Fay realized the impossibility of the operationalization [Line 307], which is also illustrated by their laughter [Lines 305 and 306]. Instead, Fay had to revise her intentional aspect by setting the subgoal of selecting values for which they could tell if the function was differentiable.

Working with this task, the students' schemes took shape through the considerations of the split domain of the function and the calculations of the derivatives for some of these values, rather spontaneously chosen. The intention was to try for the given values of the domain [Line 307], applying the rule-of-action *to determine differentiability for each of them*, starting with $x = -1$ [Line 308]. Based on the CAS feedback that the function was not differentiable for $x = -1$ or $x = 10$, the students inferred that the function was not differentiable [Fig. 10]. Making sure $x = -1$ and $x = 10$ was included in the domain, the inference of non-differentiability indicates an unarticulated theorem-in-action based on the logical deduction that *if there is a point within the domain for which the function is not differentiable, then the entire function is non-differentiable*. Still, they showed hesitation and were not sure of their solution strategy [Lines 309–311]. The students decided to move on to the following task, which is not presented in this paper. There they were to construct a non-differentiable function themselves. This created new explicit questions of their doubts [Line 312] which made them return to Task 8a.

312 E:"Construct a function which is not differentiable." [Reading aloud the task]. That I cannot do.

313 F:But I don't understand. Is there not always at least one point that will be differentiable?

314 E:Yes, I believe so. So, you cannot say if a function is differentiable.

315 F:No, but then again, I guess you should be able to, otherwise I do not think there would be such a question. And that is the same as this question [Fay scrolled back to Task 8a]. I don't understand this, whether a whole function is differentiable.

$$f1(x) \blacktriangleright \begin{cases} 4\cdot x+2, & -1\le x\le 0 \\ -0.5\cdot x^2+4\cdot x+2, & 0<x\le 4 \\ 0.1\cdot x+9.6, & 4<x\le 10 \end{cases}$$

$$\frac{d}{dx}(f1(x))|x=-1 \blacktriangleright \text{undef}$$

$$\frac{d}{dx}(f1(x))|x=10 \blacktriangleright \text{undef}$$

Funktionen er ikke diffrentiabel i -1 og 10, derfor er den ikke diffrentiabel i hele funktionen.

Fig. 10. The Em and Fay's answer to the Task 8a, copied from their written document and their written answer. The text translates to "The function is not differentiable in -1 and 10, therefore, it is not differentiable in the entire interval".

Considering Task 8a again, the students asked the teacher for a method to check if a function is differentiable. As part of this guidance, the teacher referred to the ‘for all’ statement in the definition. However, Em was still not sure of their approach to find such values for which the function could be non-differentiable [Lines 317 and 324].

316 T:Well, if you have found out there is a point, where it is not differentiable, then…

317 E:[Interrupting] Okay, so you just have to try. We were just lucky.

318 T:Lucky?

319 F:No, well, we just went with the end values.

320 T:Yes, you have to be careful with that.

321 F:Yup [nodding].

From here, the teacher made them consider the function and its graph systematically, estimating which value could be possible for the function to be non-differentiable. Looking at the graph, the students seemed to see a critical point around the red circle marked in Fig. 11.

322 E:Okay, so try for $x = 3$.

323 F:Yes, but there it was. Didn’t we try that before? [Fay typed it into TI-nspire again and got the output 1].

324 E:Yeah, okay. Then it is some crazy 3 point something number… Try with 4.

Fay typed in the derivative for $x = 4$ in TI-nspire and got the output ‘undef’.

325 F:Arh okay.

Hereby, the students found that the function was non-differentiable for $x = 4$ and could infer that the given function was non-differentiable.

In this second part of the example of Em and Fay, multiple interpretations and anticipations of ‘for all’ and its negation are present which led to the final operationalization guided by the teacher. Therefore, we first focus on Fay and afterwards Em, before we analyze their shared operationalization at the end. Fay’s question of “Is there not always one point that will be differentiable?” [Line 313] represent a theorem-in-action which does not align with the tasks they are to solve. It shows her focus on the single values known from the differentiability-scheme of a function at a point. This makes it difficult for her to consider the differentiability of a function as a concept-in-action. Considering that the students only have worked with piecewise defined functions at this stage, Fay’s theorem-in-action makes sense, as at least one point has been differentiable in the examples. The question is mathematically legitimate with an expression of the existence quantifier in the sense of MQ1, meaning at least one. It seems Fay is trying, but struggling, to imagine a function that is nowhere differentiable. This implies the inference that *a function is non-differentiable if all points of the function are non-differentiable*, which is a common misinterpretation of negating a ‘for all’ statement (Ye & Czarnocha, 2012).

Em’s comment “you cannot say if a function is differentiable” [Line 314] indicates that a function then can be both differentiable and non-differentiable, since it can include points of both kinds. This perception of the ‘for all’ quantifier is aligned with MQ4 and the interpretation that the statement is sometimes true (for some values) and sometimes not (for other values) (Sellers et al., 2021). Both Em’s and Fay’s utterances point to a concept-in-action *that differentiability is a property of a function at a single point.* At the time, the students had only worked with pointwise differentiability using the CAS-command, on which they based their interpretation of the ‘for all’ statement as meaning to check ‘for each case’. This is a different interpretation than the collective meaning of ‘for all’ (Vroom, 2022), as seen in operationalization 2. The students’ issues with Task 8a come from the discrepancy between the students’ concept-of-action of differentiability being a pointwise property and their interpretation of MQ3 as checking differentiability for every value. Thus, considering differentiability of an entire function becomes impossible and thereby, meaningless to the students, which implied the operationalization of MQ4 to check for spontaneously chosen values. After the teacher guided them towards the mathematical correct negation of finding at least one point for which the function is not differentiable [Line 316], Em was still not sure of their rule-of-action to find such a point [Line 317]. Her expression of “luck” in the operationalization of MQ4 suggests that she recognized the insufficiency of spontaneously choosing values to determine differentiability and illustrates the two issues of simultaneously dealing with the universal quantifier and the notion of differentiability.

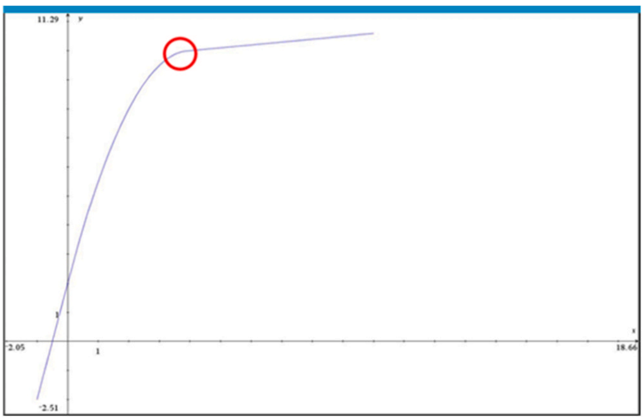

Fig. 11. The graph of the function in Task 8a. The red circle illustrates the point where Em and Fay seemed to observe a critical point.

In this case, the students' challenges with determining differentiability was, at first, to examine differentiability for an entire function, since they thought of differentiability as a property of a function for each point [Lines 301, 302, 313, and 314]; second, the impossible task of examining the entire function for each point [Lines 305–306]; third, to negate the 'for all' statement; and forth, deliberately to choose meaningful values to examine the function [Line 317]. Not until they consulted the graph, getting that the function was differentiable for $x = 3$ which they already knew, they were guided towards trying for the value $x = 4$ [Lines 322–324], still rather randomly.

5. Discussion

Recalling our research question, i.e., students' ways of operationalizing the 'for all' statement of differentiability when using CAS, we now turn to the discussion of our findings. In the following discussion, we consider three key areas: (1) the relationship between students' operationalizations and the affordances and constraints of CAS; (2) the influence of the Danish educational context and the specific education design in shaping both the learning environment and students' interpretative practices; and (3) the theoretical and methodical choices.

5.1. The role of CAS for operationalizations of 'for all'

The analyses of students' work illustrate that their anticipations often, but not always reflect a mathematically correct interpretation. The challenges with solving the tasks appear to stem from operationalizing these anticipations with CAS. The three cases illustrate how the students' use of CAS to operationalize the 'for all' statement of differentiability relies on their anticipations and capabilities of turning their interpretations into instrumented techniques for determining differentiability of a function. This means that a mathematically correct interpretation of a statement may not be enough for being able to solve a given task (based on that statement). Equivalently, a student's mathematical wrong interpretation of a statement may not be enough to understand their operationalizations of and difficulties with that statement. Hence, we find two main challenges to when working with mathematical statements using CAS. The first is to understand and interpret the statement mathematically correctly. The other is to operationalize these interpretations through CAS commands. The analyses illustrate the relation between the students' interpretations of mathematical statements as found likewise in other research (e.g., Epp, 2009; Piatek-Jimenez, 2010; Ye & Czarnocha, 2012) and their mental actions of quantifying variables (Sellers et al., 2021) actualized into instrumented techniques and actions.

We identified three ways of operationalizing the 'for all' statement of examining differentiability of a function, where CAS played a central role in supporting or challenging these operationalizations. Operationalization 1 was to use the differentiability-scheme to detect and eliminate counterexamples. Here, students worked with existence as MQ1 (Sellers et al., 2021) by checking for at least one counterexample, as it being the negation of 'for all'. The approach mirrored our intentions of the differentiability-scheme by applying the derivative-technique to eliminate or confirm counterexamples. This builds on the logic that the given function passes the test of the definition if no counterexamples exist (Epp, 2009; Yopp, 2020). Operationalization 2 was to extend the derivative-technique to check for all values at once as a collective set, and operationalization 3 was to apply the differentiability-scheme for every value one at the time. Both operationalizations 2 and 3 build on the same meaning of the 'for all' statement, defined as MQ3 to check for all values (Sellers et al., 2021). Again, this interpretation is mathematically correct, but the students developed different schemes for tackling this check. In operationalization 2, the students anticipated to realize this interpretation by using the derivative-technique to check for an interval as a collective set, identified as MQ5 (Sellers, 2020). Operationalization 3 involved the anticipation of literary checking differentiability for every value of the domain. The impossibility of this task implied a changed anticipation of checking for spontaneously chosen values, identified as MQ4 (Sellers et al., 2021).

For the operationalizations 1 and 3, CAS' derivative-command can support an anticipation of non-differentiability of a function by detecting counterexamples or challenge that same anticipation when eliminating counterexamples. Thereby, CAS can be rather unsupportive of operationalizing the universal quantifier as it only provides single examples, but supportive of operationalizing the existence quantifier. For operationalization 2, CAS can be helpful both in case of differentiability and non-differentiability, because the output is the entire derivative including the domain for which the derivative is defined. All three operationalizations require that the students are familiar with the domain of a function and piecewise defined functions as well as recognizing that the end values are uninteresting for determining differentiability. These are important elements when operationalizing the 'for all' quantifier with which CAS have not shown to be supportive. CAS drew no attention to the importance of the domains of the given functions and how these were defined. However, CAS may challenge the students' anticipations, which can cause further curiosity and investigations. The limitations of these findings are related to the specific choice of TI-nspire CAS, which has its own syntax and configurations of logic. Other tools, such as Maple may facilitate abstract algebra better, and Volfram Alpha or more recently, large language models relate differently to natural language, logic and calculus. The choice of TI-nspire was due to the local context of the study, which was conducted at a time where TI-nspire and similar CAS programs were extremely integrated in Danish mathematics curriculum.

5.2. Revisiting the educational context and the task design in light of students' challenges

The study reports upon several challenges that the students meet during their work of differentiability. These challenges may be related to the mathematical content involved (e.g., Roh, 2010; Zandieh, 2000), induced by the digital tools (e.g., Jankvist & Misfeldt, 2015; Winsløw, 2003), as also seen above, and due to the design and formulations of the tasks. The tasks were designed for the specific project of investigating the interplay between mathematical thinking competency of the KOM framework and digital technologies

(Pedersen, 2024). This made the tasks quite unlike typical mathematics tasks in Danish upper secondary school. The extreme application of CAS in the tasks, on the other hand, was similar to how CAS was applied at upper secondary level in general. The students' way of consulting CAS and the graphic view, observing and tracing the graphs of the given functions can be considered rather typical for the time in a Danish mathematics classroom.

All functions in the task sequence are piecewise defined, and therefore, at least piecewise differentiable. The analysis shows that this evokes certain concepts- and theorems-in-action. For instance, Lou presumed all piecewise functions to be non-differentiable. Moreover, explicitly asking students to use the derivative-command in CAS for specific values, as in Tasks 6a and 7a, may have influenced their challenges of viewing differentiability as a property of the entire function, as seen with Group 5. This approach may also have affected how they assessed the functions given. Some students, as seen in Group 11 consulted the graph superficially without looking at tangents or secants. Yet, none of the students used the graphical representation of the tangent on the graph to empirically run over the function as an approximation to check every value in the interval, despite the students having worked with the graphic representations of differentiability at point, previously. The auxiliary tasks (Tasks 6a and 7a) were thought to be part of the generative aspect of their schemes, guiding the attention towards the domain of the function and the critical values as important for determining differentiability. Instead, this turned out to be a main issue for the students. For pragmatic reasons all functions were defined in the premade TI-nspire document given to the students. Thus, the students were not typing the domain themselves and therefore may not have been aware of the domain or the critical values. Letting them type in the functions themselves could help them pay attention to the composition of the function and for which values it was defined. Whether this would be a solution to this issue is unclear, and easily many other syntax errors could occur. Other solutions could be to add different kinds of auxiliary tasks, for instance, between Task 6a and 6b to point the attention directly to the choice of the values for investigation or by introducing piecewise defined functions for itself prior to these tasks.

The configurations of the tasks inevitably influence the students. Reformulating the tasks in different ways may affect the students' interpretations and anticipations in other ways. In general, it is important as a teacher to be aware of these influences. More importantly, the study indicates that the teacher should be aware of the students' theorems-in-action, as these are crucial for their way of thinking and can serve as steppingstones for their operationalizations of the quantifiers. Students may have many correct interpretations and anticipations of the universal quantifier, which are important to share in the classroom and with the teacher. In the three cases, the students worked from different theorems-in-action, but the teacher was rarely aware of these, such as Hal and Gia's extension of the derivative-technique applied to an interval, or Em and Lou questioning differentiability of a whole function. When the students asked for guidance, the teacher unintentionally directed the students in a different direction than their previous attempt, which not necessarily supported or built upon the students' own anticipations. In the case of Kay and Lou, the students shared their theorem-in-action of an existing non-differentiable point because of the function's symbol expression and the teacher could guide them from their point of departure. This is in line with teacher practices that facilitate students' discussions to engage with structures of mathematical definitions (Park et al., 2024). In this regard, it would be interesting to focus more on the student-teaching interaction for further studies of students' operationalizations of quantifiers. It could be valuable to study a teaching that encourages the students to make their interpretations and anticipations explicit to each other and the teacher. The teacher, in turn, can be curious, ask the students about their process, and use their work as a starting point for further exploration. Thus, focusing on structures of mathematical statements in teaching and learning situations may require a teaching approach that focuses more on the students' process than on right and wrong solutions to a mathematical task. Nonetheless, a limitation of the study is that it only focuses on students' cognitive processes of solving tasks with CAS and not on social dynamics and student-teacher interactions. This is, of course, an important perspective of enunciation of quantifiers in the classroom.

The epistemic gap between working with differentiability of a function *at one point* using CAS and considering differentiability of *an entire function* rests on the ability to abstract the concept of differentiability to be a property of a function. Abstracting and relating concepts to different domains and contexts are also part of the mathematical thinking competency. Through such work, the students hypothesize (explicitly or unarticulated) about the concepts involved and their relations, as also illustrated in our analyses. These hypotheses, being part of their mathematical thinking, can be identified through the lens of Vergnaud's (2009) schemes, particularly the construct of theorems-in-action, also in situations where CAS are absent. Thus, working explicitly with students' hypothesis in terms of their theorems-in-action of the definition of differentiability can focus the attention on the structure of mathematical statements and the logic behind the included quantifiers as well as the extension from pointwise, piecewise, and global differentiability. Thereby, the topic of differentiability has potential for working with and developing these aspects of the mathematical thinking competency. However, other aspects of this competency, such as working with mathematical questions and answers, may not be touched upon through this work.

5.3. The role of the theoretical and methodological perspectives

For the analysis of the students' work, instrumental genesis brought forward the students' instrumented techniques which were a driving factor of their operationalizations, as seen in the case of Hal and Gia who literally extended the instrumented techniques of determining differentiability of a point to an interval by changing the specifications. To elaborate operationalizations, the four aspects of the scheme helped indicate the rationale behind them. For instance, the use of CAS influenced the generative aspect of the students' schemes by making them change their rules-of-actions. This led to questioning and revising the intentional, epistemic and computational aspects of operationalizing the 'for all' quantifier, either by the students themselves or through guidance from the teacher. In the meeting between the students' schemes and the feedback from CAS, we could identify challenges that the students encountered, and the notion of scheme and its four aspects helped explain why these challenges occurred.

The distinctions between goals, anticipations, rules-of-action, theorems-in-actions, inferences and techniques worked as a pair of lenses to distinguish the students' interpretations, anticipations and actual operationalization, elaborated by meanings of quantification (Sellers et al., 2021), eliminating counterexamples (Yopp, 2020), and previous findings on students' interpretations of mathematical statements and quantifiers (e.g., Ye & Czarnocha, 2012). We identified possible agreements and conflicts between students' anticipations, theorems-in-action and instrumented techniques using CAS. Their anticipations regarded how to check for differentiability, what commands to use, what output this would provide and how that would align with their interpretation of the 'for all' statement. These anticipations were built on their theorems-in-action about differentiability and non-differentiability. The students' techniques were then determined by the possibilities of using and decoding the CAS commands and outcomes. These findings are based on a small amount of qualitative data illustrating single empirical cases and are not intended to be statistically generalizable to a broader population. They illustrate examples of upper secondary students' operationalizations of the 'for all' quantifier that are similar across the students of the study. The anticipations on which the students build their operationalization are similar to findings of previous research (e.g., Sellers et al., 2021; Ye & Czarnocha, 2021). Using the technique-scheme duality and the four aspects bring forward that the interpretations of 'for all' and its negation we know from undergrad and tertiary level also apply for upper secondary school, which makes sense because university students build their interpretations on previous experiences. Focusing on teaching quantifiers in simpler settings in upper secondary school may influence how they interpret them at university level. However, these findings only illustrate how students with access to CAS handled the situations, and not whether students in general would have similar operationalization of the universal quantifier. The context of using CAS illustrates how the interpretations are expressed when upper secondary students work with CAS, and how this may imply, support or challenge these interpretations. Distinguishing between anticipations and operationalizations elaborate how the anticipations may be either operationalized in different ways or challenge the students to operationalize the universal quantifier.

6. Conclusion

Through a series of task designs, drawing on theoretical elements from the instrumental approach, i.e., the scheme-technique duality and Vergnaud's notion of scheme we have addressed the question of what ways do upper secondary students operationalize the 'for all' statement in the definition of differentiability of a function when using CAS. We have identified three ways of operationalizations: (1) Using the differentiability-scheme to detect and eliminate counterexamples, (2) extending the derivative-technique to check for all values at once, and (3) applying the differentiability-scheme for every value. The three ways of operationalizing the 'for all' statement of differentiability of a function illustrate how students can be challenged by interpreting the universal quantifier, anticipating how to turn their interpretations into actions, and to carry out these actions using CAS. Whether they have interpreted the universal quantifier in an either correct or wrong way, they may not necessarily have the techniques to operationalize it. Hence, students' interpretations, anticipations, and operationalizations can be considerable distinct. Our analyses indicate that the main idea of Operationalization 1 (to find values of possible counterexamples) is significant for applying different techniques in a meaningful manner. Hence, decoding and exploiting the domain of the piecewise functions is essential for solving these kinds of tasks. However, neither the formulations of the tasks nor the application of CAS pointed the attention to the domain of the functions. First, when the students managed to decode and exploit the split domain and the critical values, CAS could challenge or confirm the students' anticipations, assisting them in the process of determining differentiability. The study indicates that the teacher approach should be curious of the students' theorems-in-action and encourage them to investigate their own operationalizations. Focusing on the 'for all' statement in the definition of differentiability of a function can serve as an entry for upper secondary mathematics students to work with logical structures of mathematical statements and thereby put mathematical thinking competency into action in this regard.

CRediT authorship contribution statement

Uffe Thomas Jankvist: Writing – review & editing, Writing – original draft, Validation, Supervision, Funding acquisition, Conceptualization. **Morten Misfeldt:** Writing – review & editing, Writing – original draft, Validation, Supervision, Conceptualization. **Mathilde Kjær Pedersen:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used OpenAI/ChatGPT in order to improve language, avoid wordiness and sentence fragments. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

Declaration of Competing Interest

he authors declare that there are no conflicts of interests.

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